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Duality Symmetries from Non–Abelian Isometries in String Theory*

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In string theory it is known that abelian isometries in the σ–model lead to target space duality. We generalize this duality to backgrounds with *non–abelian* isometries. The procedure we follow consists of gauging the isometries of the original action and constraining the field strength F to vanish. This new action generates dual theories by integrating over either the Lagrange multipliers that set $F = 0$ or the gauge fields. We find that this new duality transformation maps spaces with non–abelian isometries to spaces that may have no isometries at all. This suggests that duality symmetries in string theories need to be understood in a more general context without regard to the existence of continuous isometries on the target space (this is also indicated by the existence of duality in string compactifications on Calabi–Yau manifolds which have no continuous isometries). Physically interesting examples to which our formalism apply are the Schwarzschild metric and the 4D charged dilatonic black hole. For these spherically symmetric black holes in four dimensions, the dual backgrounds are presented and explicitly shown to be new solutions of the leading order string equations. Some of these new backgrounds are found to have no continuous isometries (except for time translations) and also have naked singularities.

10/92

1. Introduction

One of the most interesting properties discovered in nontrivial string backgrounds is the existence of target space duality [1]. Besides providing a better understanding of the moduli space of a given solution, it may lead to interesting cosmological consequences as emphasized in [2]. It can also provide crucial information about the low energy couplings in string compactifications, by requiring the interactions to be expressed in terms of the modular functions of a 'duality' group [3].

Duality symmetries were originally discovered for toroidal compactifications of closed string theories [1]. Subsequently, it has been realized that they are a property of all string vacua with abelian isometries [4] and the invariance of the partition function under a duality transformation was shown for the corresponding σ–model. In this way, duality on curved string backgrounds such as 2D black holes or more general gauged WZW models has been understood [5,6,7,8]. This however cannot be the end of the story since similar symmetries exist for compactifications on Calabi–Yau spaces [9], even though these are string backgrounds without isometries*. At the moment there is no understanding of these symmetries in terms of the σ–model action.

In this paper we address the question of whether there exist duality transformations for string backgrounds with non–abelian isometries. The physical motivation is clear. The discovery of these duality symmetries represents another step in the classification of physically inequivalent string vacua. It is for example very interesting to study solutions of Einstein's equations in vacuum since they are also solutions of the leading order string background equations with constant dilaton and antisymmetric tensor field. It also happens that *every* known solution of Einstein's equations in 4D has isometries [10]. Some of the most interesting 4D geometries, such as the Schwarzschild solution and the Friedman–Robertson–Walker (FRW) homogeneous cosmologies, have non–abelian isometries: they have an $SO(3)$ spherical symmetry.

Given a σ–model action with a global symmetry, the procedure we follow starts by gauging the symmetry in the σ–model action and adding to it an extra term with a Lagrange multiplier which constrains the gauge field strength to vanish. After integrating over the Lagrange multiplier and fixing the gauge, we recover the original action. On the

* Duality symmetries for Calabi–Yau compactifications are analogous to duality symmetries for toroidal compactifications. They are the symmetries in the modular group of the Kähler class parameter space of the Calabi–Yau metric and relate "large" with "small" manifolds.

other hand, by integrating by parts the Lagrange multiplier term and then integrating out the gauge fields, we obtain the dual action in which the Lagrange multiplier is a new dynamical field. This procedure is equivalent to the standard first order formalism used for abelian isometries [4,8] as emphasized recently in [11,12], but it generalizes more readily to the non–abelian case*. As we will see, the question of whether there exist duality symmetries, can be posed for any worldsheet σ–model with target space isometries, regardless of the conformal invariance of the theory. Conformal invariance is however mantained by also transforming the dilaton field appropriately [15,4].

Probably the most intriguing property of this new duality that we have found is that it can map a geometry with non–abelian isometries to another which has none. This is remarkable because starting from the geometry with no isometries, the current procedure for performing duality transformations would not give any information about the existence of the 'dual' geometry. Nevertheless the two geometries are indeed related and even though they are very different as geometries, they give the same partition function. This indicates that there should be a more general argument, deeper and independent of the existence of continuous isometries, by means of which duality transformations can be explained. As mentioned above, this is also implied by the existence of duality–like symmetries in Calabi–Yau spaces.

To set up conventions and notation, as well as to make the paper self contained, in Sect. 2 we review briefly the duality transformation for the case of abelian isometries of the target space. In Sect. 3 we present the generalization to non-abelian isometries. We discuss in detail the change in the dilaton field necessary to render the theory conformally invariant to first order. We give the duality transformation for $SO(N)$ and present it explicitly for $SO(3)$, which is the case relevant to 4D spherically symmetric solutions. In Sect. 4 we find the dual geometries of Schwarzschild and of charged dilatonic black holes, for which the isometry group is $SO(3) \times \{\text{time translations}\}$. We make concluding remarks and discuss our results in Sect. 5.

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* For a related discussion of duality in terms of the exchange of Bianchi identities and field equations of 2D σ–models on group manifolds, see [13]. For a review of earlier work on duality in field theories with non–abelian symmetries see [14].

2. Duality with Respect to Abelian Isometries

Target space duality is a general property of string vacua which have at least one isometry. We will briefly review here the case where the isometries are abelian in order to be able to understand the generalization to non–abelian isometries. In ref. [4], the '$r \to 1/r$' duality in toroidal compactifications of string theories was generalized to any string background for which the metric in the worldsheet action had at least one isometry. Examples where this duality transformation has been applied are the 2D black-hole and some simple cosmological string backgrounds. The worldsheet action for the bosonic string in a background with N commuting isometries, can be written as

$$
\begin{aligned}
S = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2 z \Big(& Q_{\mu\nu}(X_\alpha)\, \partial X^\mu \overline{\partial} X^\nu + Q_{\mu n}(X_\alpha) \partial X^\mu \overline{\partial} X^n \\
& + Q_{n\mu}(X_\alpha) \partial X^n \overline{\partial} X^\mu + Q_{mn}(X_\alpha) \partial X^m \overline{\partial} X^n \\
& + \alpha' R^{(2)} \Phi(X_\alpha) \Big) \ ,
\end{aligned}
\tag{2.1}
$$

where $Q_{MN} \equiv G_{MN} + B_{MN}$ and lower case latin indices m, n label the isometry directions. Since the action (2.1) depends on the X^m only through their derivatives, we can write it in first order form by introducing variables A^m and adding an extra term to the action $\Lambda_m(\partial \overline{A}^m - \overline{\partial} A^m)$ which imposes the constraint $A^m = \partial X^m$. Integrating over the Lagrange multipliers Λ_m returns us to the original action (2.1). On the other hand performing partial integration and solving for A^m and $\overline{A}^m$, we find the dual action S' which has an identical form to S but with the dual background given by [4,8]

$$
\begin{aligned}
Q'_{mn} &= (Q^{-1})_{mn} \\
Q'_{\mu\nu} &= Q_{\mu\nu} - Q_{\mu m}\, (Q^{-1})^{mn}\, Q_{n\nu} \\
Q'_{n\mu} &= (Q^{-1})_n^{\ m}\, Q_{m\mu} \\
Q'_{\mu n} &= -Q_{\mu m}\, (Q^{-1})^m_{\ n} \ .
\end{aligned}
\tag{2.2}
$$

To preserve conformal invariance, it can be seen [4,15] that the dilaton field has to transform as $\Phi' = \Phi - \log \det G_{mn}$. Notice that equations (2.2) reduce to the usual duality transformations for the toroidal compactifications of [16] in the case $Q_{m\mu} = Q_{\mu m} = 0$ and can map a space with no torsion ($Q_{m\mu} = Q_{\mu m}$) to a space with torsion ($Q'_{m\mu} = -Q'_{\mu m}$). For the case of a single isometry, we recover the explicit expressions of [4].

It is not necessary to go to the first order formalism for every isometry direction since we do not have to perform a duality transformation for all of them. That is, we can integrate over a subset of the Lagrange multipliers Λ_m and, for the remaining isometry directions, integrate out the corresponding gauge fields A^m. Equation (2.2) should then be read with the indices m, n running only over the variables with isometries that have been dualized. The total duality group includes these transformations as well as shifts in the antisymmetric tensor field and has been argued [17,18,12] to be equivalent to $SO(N, N, \mathbb{Z})$.

An equivalent interpretation of the duality process just described is given in [11]. In the original action the symmetry is gauged by replacing ∂X^m with $DX^m = \partial X^m + A^m$ and the term $\int d^2z\ \Lambda_m(\partial\overline{A}^m - \overline{\partial}A^m)$ is added to the action. This extra term imposes the vanishing of the field strength F of the gauge fields after integration over the Lagrange multipliers Λ_m. This implies that the gauge field must be pure gauge, $A^m = \partial\widetilde{X}^m$. The gauge fixing can be done either by choosing the gauge fields to vanish or by taking $X^m = 0$ (a unitary gauge). In both cases this reproduces the original action. The dual theory is obtained by instead integrating out the gauge fields and then fixing the gauge. This is the procedure that generalizes the duality transformation to the case of non–abelian isometries.

3. Duality with Respect to Non–Abelian Isometries

Consider the σ–model action (2.1) and assume that the target space metric has a group $\mathcal{G}$ of non–abelian isometries. In this case, Q_{MN} *does* depend on X^m and transforms accordingly under $X^m \rightarrow g^m_{\ n}X^n$, $g \in \mathcal{G}$. We now follow the procedure of Ref[11]. We gauge the symmetry corresponding to a subgroup $H \subseteq \mathcal{G}$

$$\partial X^m \rightarrow DX^m = \partial X^m + A^\alpha (T_\alpha)^m_{\ n} X^n \ , \tag{3.1}$$

and add to the action the term

$$\int d^2z\ tr(\Lambda F) = \int d^2z\ \Lambda_\alpha F^\alpha \ , \tag{3.2}$$

where in this case the gauge field strength is, in matrix notation,

$$F = \partial\overline{A} - \overline{\partial}A + [A, \overline{A}] \ . \tag{3.3}$$

The $N \times N$ matrices T_α form an adjoint representation of the group H, $[T_\alpha, T_\beta] = c_{\alpha\beta}^{\ \ \gamma} T_\gamma$, normalized such that $tr(T_\alpha T_\beta) = \delta_{\alpha\beta}$ (a constant in the normalization can be absorbed in a redefinition of the Lagrange multiplier). In the path integral we have then

$$\begin{aligned} \mathcal{P} &= \int \mathcal{D}X\ e^{-iS[X]} \\ &= \int \mathcal{D}X \int D\Lambda \int \frac{DAD\overline{A}}{V_\mathcal{G}}\ \exp\left\{ -i\left(S_{gauged}[X, A, \overline{A}] + \int d^2z tr(\Lambda F) \right) \right\} \ , \end{aligned} \tag{3.4}$$

where $V_{\mathcal{G}}$ is the "volume" of the group of isometries and $\mathcal{D}X$ is the measure that gives the correct volume element

$$\mathcal{D}X = DX\sqrt{G}e^{-\Phi} \ . \tag{3.5}$$

Similar to the abelian case, the original action is obtained by integrating out the Lagrange multiplier Λ. Locally, this forces the gauge field to be pure gauge

$$A = h^{-1}\partial h \ , \quad \overline{A} = h^{-1}\overline{\partial} h \ , \qquad h \in H \ . \tag{3.6}$$

By fixing the gauge with the choice $A = 0$, $\overline{A} = 0$ we reproduce the original theory. A different gauge choice will only give the same theory in a different coordinate system.

The dual theory is obtained by integrating out the gauge fields in the path integral (3.4). Integrating over the gauge fields $\overline{A}$ we obtain

$$\mathcal{P} = \int \frac{\mathcal{D}X}{V_{\mathcal{G}}}\ D\Lambda \int DA\ \delta\left(fA + h\right)\ \exp\left\{-i\left(S[X] + \frac{1}{4\pi\alpha'}\int d^2z\ \overline{h}_\alpha A^\alpha\right)\right\} \ , \tag{3.7}$$

where $S[X]$ is the original action and h, $\overline{h}$ and the matrix f are given by

$$\begin{aligned} h_\alpha &= -\partial\Lambda_\alpha + \left(Q_{\mu n}\partial X^\mu + Q_{kn}\partial X^k\right)(T_\alpha)^n{}_m X^m \ , \\ \overline{h}_\alpha &= \ \ \overline{\partial}\Lambda_\alpha + \left(Q_{n\mu}\overline{\partial} X^\mu + Q_{nk}\overline{\partial} X^k\right)(T_\alpha)^n{}_m X^m \ , \\ f_{\alpha\beta} &= -c_{\alpha\beta}{}^\gamma \Lambda_\gamma + X^k (T_\beta)^j{}_k Q_{jn} (T_\alpha)^n{}_m X^m \ . \end{aligned} \tag{3.8}$$

Integrating out the gauge field A we thus obtain

$$\mathcal{P} = \int \frac{\mathcal{D}X}{V_{\mathcal{G}}}\ D\Lambda\ e^{-iS'[X,\Lambda]} \det(f^{-1}) \ , \tag{3.9}$$

with $S'[X,\Lambda]$ given by

$$S'[X,\Lambda] = S[X] - \frac{1}{4\pi\alpha'}\int d^2z\ \overline{h}_\alpha (f^{-1})^{\alpha\beta} h_\beta \ . \tag{3.10}$$

We should now fix the gauge to eliminate the extra degrees of freedom that were introduced by gauging the original action. To do this we use the gauge symmetry to introduce a convenient gauge: $\widehat{X} = h_0 X$, $\hat{\Lambda} = h_0^{-1}\Lambda h_0$, for a given $h_0 \in H$. Obviously, different gauge choices *will not* give different dual theories, they will give the same theory differing by a coordinate transformation. Notice that the maximum number of Lagrange multipliers that can be gauged away is $\dim\mathcal{G} - \mathrm{rank}\mathcal{G}$. Indeed, the number of Lagrange multipliers invariant under the group of isometries is rank$\mathcal{G}$. Using the Fadeev–Popov method to fix the gauge in the path integral we obtain

$$\mathcal{P} = \int \mathcal{D}X \; D\Lambda \; \delta[\mathcal{F}] \; \det \frac{\delta \mathcal{F}}{\delta \omega} \; e^{-iS'[X,\Lambda]} \det(f^{-1}) \; , \tag{3.11}$$

where $\mathcal{F}$ is the gauge fixing function and ω are the parameters of the group of isometries. Therefore

$$\mathcal{P} = \int \mathcal{D}Y \; e^{-iS'[Y]} \det(f(Y)^{-1}) \; . \tag{3.12}$$

Here we have denoted the new coordinates $\widehat{X}$ and $\hat{\Lambda}$ on the dual manifold collectively by Y. The Fadeev–Popov determinant in the path integral contributes to the measure such that the correct volume element for the dual manifold is *precisely* obtained

$$\mathcal{D}Y = DY \sqrt{G'} e^{-\Phi'} \; . \tag{3.13}$$

The factor $\det(f^{-1})$ in the partition function can be computed using standard heat kernel regularization techniques (see for example references [4], [7] and [19]). It generates a new local term in the action of the form

$$\frac{1}{4\pi\alpha'} \int d^2z \; \alpha' R^{(2)} \; (\Delta\Phi) \; , \tag{3.14}$$

which corresponds to the change in the dilaton due to the duality transformation

$$\Phi' = \Phi - \log\det f \; . \tag{3.15}$$

This change in the dilaton transformation is the shift necessary to retain the conformal invariance of the dual theory. There is actually another prescription from which this change in the dilaton can be obtained. In fact, the requirement that the correct volume element (3.13) is obtained in the dual theory means that

$$e^{-\Phi'} = \left[e^{-\Phi} \sqrt{\frac{G}{G'}} \; \det \frac{\delta \mathcal{F}}{\delta \omega} \right]_{\mathcal{F}=0} \; , \tag{3.16}$$

as can be checked from equations (3.11) and (3.13). This prescription coincides with the prescription for duality with respect to abelian isometries in [4] because in this case the Fadeev–Popov determinant is trivial. A consistency check of the change in the dilaton is obtained by comparing equations (3.16) and (3.15).

In general, we cannot write explicitly the gauge fixed dual action. Therefore, we are not able to present the new metric and antisymmetric tensor fields in a closed form, as was done for the abelian case in equations (2.2). In the following we shall study a number of examples and in these cases the gauge fixing procedure will be carried through in detail. This will allow us to have explicit expressions for the dual background fields. As an example to which we will extensively refer, let us consider a theory for which the target space metric has a maximally symmetric subspace with $\mathcal{G} = SO(N)$ and no antisymmetric tensor. The coordinates $X^M, M = 1, ..., D$, can be decomposed into $N-1$ angular coordinates (θ^i) describing $(N-1)$–dimensional spheres, and $D-N+1$ extra coordinates (v^μ) specifying the different spheres in the D dimensional spacetime. The metric can then be decomposed as [20] in the form

$$ds^2 = g_{\mu\nu}(v)dv^\mu dv^\nu + \Omega(v)g_{ij}d\theta^i d\theta^j \ . \tag{3.17}$$

It is more convenient to treat the coordinates θ^i in terms of cartesian coordinates X^m in a N dimensional space on which $SO(N)$ can act linearly, so we write the σ model action in the form

$$\begin{aligned} S[v,X] = S[v] + \int d^2z\ \Omega(v)\left\{g_{mn}\partial X^m\overline{\partial}X^n + \frac{1}{2a\sqrt{\Omega}}\lambda(g_{mn}X^mX^n - a^2)\right\} \\ + \frac{1}{4\pi\alpha'}\int d^2z\ \alpha' R^{(2)}\ \Phi\ , \end{aligned} \tag{3.18}$$

where $S[v] = \int d^2z g_{\mu\nu}(v)\partial v^\mu\overline{\partial}v^\nu$, the metric g_{mn} is diagonal and constant and the Lagrange multiplier term fixes the N dimensional space to be a sphere of radius a. The factor $\frac{1}{2a\sqrt{\Omega}}$ has been introduced to obtain the correct volume element of the sphere after integrating over λ. Gauging this action (with a vanishing field strength) and fixing the gauge $A = \bar{A} = 0$ we obtain the original action. To find the dual action we can take antisymmetric matrices T_α and h, $\overline{h}$ and f are given by

$$\begin{aligned} h_\alpha &= -\partial\Lambda_\alpha + \Omega(v)\partial X^n(T_\alpha)_{nm}X^m\ , \\ \overline{h}_\alpha &= \ \ \overline{\partial}\Lambda_\alpha + \Omega(v)\overline{\partial}X^n(T_\alpha)_{nm}X^m\ , \\ f_{\alpha\beta} &= -c_{\alpha\beta}{}^\gamma\Lambda_\gamma - \frac{1}{2}\Omega(v)X^n\left\{T_\beta, T_\alpha\right\}_{nm}X^m\ . \end{aligned} \tag{3.19}$$

A convenient choice of gauge is to set

$$X^m = 0, \quad m = 1, ..., N-1, \quad X^N = a \tag{3.20}$$

and then use the remaining $SO(N-1)$ gauge freedom (the gauge transformations that preserve (3.20)) to gauge away $\frac{1}{2}(N-1)(N-2)$ of the Lagrange multipliers Λ_α. We then obtain

$$
\begin{aligned}
h_\alpha &= -\partial\Lambda_\alpha \ , \\
\overline{h}_\alpha &= \ \ \overline{\partial}\Lambda_\alpha \ , \\
f_{\alpha\beta} &= -c_{\alpha\beta}{}^\gamma\Lambda_\gamma - a^2\Omega(v)(T_\alpha)_{Nm}(T_\beta)^m{}_N \ ,
\end{aligned}
\tag{3.21}
$$

and

$$
S^{dual}[v,\Lambda] = S[v] + \frac{1}{4\pi\alpha'}\int d^2z\ \left(\partial\Lambda_\alpha(f^{-1})^{\alpha\beta}\overline{\partial}\Lambda_\beta\right) + \frac{1}{4\pi}\int d^2z\ R^{(2)}\Phi' \ . \tag{3.22}
$$

From this expression, we can in principle read off the new background fields as in (2.2). The only problem is that we still have to complete the gauge fixing for the Λ_α. In order to be more explicit, we will now consider in greater detail examples with $\mathcal{G} = SO(2)$ and $\mathcal{G} = SO(3)$. The case for $SO(2)$, even though abelian, will be considered to fix ideas and to show how the formalism contains the abelian case.

When $\mathcal{G} = SO(2)$ there is only one matrix T

$$
T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{3.23}
$$

In the gauge $X^1 = a$, $X^2 = 0$ and $A = \partial\theta$ (that is, in spherical coordinates), the original action (3.18) reduces to

$$
S[v,\theta] = S[v] + \int \mathrm{d}^2z\ a^2\Omega(v)\partial\theta\overline{\partial}\theta \ . \tag{3.24}
$$

To obtain the dual theory we start by calculating the quantities h, $\overline{h}$ and f. Before fixing the gauge we have

$$
\begin{aligned}
f &= a^2\Omega(v) \ , \\
h &= -\partial\Lambda - a^2\Omega(v)\partial\theta \ , \\
\overline{h} &= \ \ \overline{\partial}\Lambda - a^2\Omega(v)\overline{\partial}\theta \ ,
\end{aligned}
\tag{3.25}
$$

where we have used spherical coordinates $X^1 = a\cos\theta$, $X^2 = a\sin\theta$. The action (3.10) before fixing the gauge is therefore

$$
S'[v,\Lambda] = S[v] + \frac{1}{4\pi\alpha'}\int d^2z\ \left(\frac{1}{a^2\Omega(v)}\partial\Lambda\overline{\partial}\Lambda + \partial\theta\overline{\partial}\Lambda - \partial\Lambda\overline{\partial}\theta\right) \ . \tag{3.26}
$$

After fixing the gauge by choosing $\theta = 0$, we obtain for the dual action

$$S^{dual}[v,\Lambda] = S[v] + \frac{1}{4\pi\alpha'}\int d^2z\ \left(\frac{1}{a^2\Omega(v)}\partial\Lambda\bar{\partial}\Lambda\right) + \frac{1}{4\pi}\int d^2z\ R^{(2)}\Phi'\ , \tag{3.27}$$

where $\Phi' = \Phi - \log a^2\Omega(v)$. As expected, we have recovered the '$r \to \frac{1}{r}$' duality symmetry on the circle. It is important to note here that when fixing the gauge we could not have gauged away the Lagrange multiplier Λ instead of eliminating θ because it is *invariant* under the SO(2) gauge transformations. In the dual theory, what started life as a Lagrange multiplier became a coordinate.

Note also that the dual metric (3.27) has an isometry since it depends on Λ only through its derivatives. This feature is general: if a metric has a commutative isometry associated to a coordinate φ, then the dual theory with respect to this isometry also has an isometry associated to the coordinate Λ, dual to φ, which appeared as a Lagrange multiplier in passing to the dual theory. This is easily seen by noting that when the isometry is abelian the term (3.2) is invariant under constant shifts of Λ as a consequence of partial integration. This symmetry is mantained even after integrating out the gauge fields. In fact, when the isometry is abelian, the matrix $f_{\alpha\beta}$ in (3.19) becomes completely independent of Λ since the structure constants vanish and hence, the dual action depends on Λ only through its derivatives. It is important to remark though that abelian isometries disappear under duality with respect to a group of non–abelian isometries that contains the corresponding abelian group.

We would like to remark that this formalism works also in coordinate systems in which the abelian isometries do not manifest themselves as constant shifts of the coordinates. Consider for example the 2D black hole [21] with metric in Kruskal–like coordinates

$$ds^2 = -\frac{dudv}{1-uv}\ . \tag{3.28}$$

The corresponding nonlinear σ–model does not have the form (2.1) because the metric does depend explicitly on u and v. To perform a duality transformation following [4] is therefore necessary to change coordinates so that the σ–model has the form (2.1). However, since the isometry of (3.28) manifests itself as a SO(1,1) symmetry action on the coordinates ($u \to e^{\alpha}u$ and $v \to e^{-\alpha}v$), we can follow our formalism to find the dual metric without performing first a coordinate transformation. We reproduce the fact that this background is self dual [5].

For $\mathcal{G} = SO(3)$ there are three matrices T. Due to the fact that in three dimensions an antisymmetric matrix is dual to a three–vector, we can choose $(T^p)^m_{\ n} = \epsilon^{pm}{}_n$. The original action in spherical coordinates (the gauge choice here is $X^1 = X^2 = 0$, $A^1 = \partial\theta$, $A^2 = -\sin\theta\partial\varphi$ and $A^3 = \cos\theta\partial\varphi$) is

$$S[v,\theta,\varphi] = S[v] + \int d^2z\ a^2\Omega(v)\left(\partial\theta\bar{\partial}\theta + \sin^2\theta\partial\varphi\bar{\partial}\varphi\right) \ . \tag{3.29}$$

We now find the dual theory. Before fixing the gauge

$$\begin{aligned} h_p &= -\partial\Lambda_p + \Omega(v)\partial X^m\epsilon_{pmn}X^n \ , \\ \bar{h}_p &= \quad \bar{\partial}\Lambda_p + \Omega(v)\bar{\partial}X^m\epsilon_{pmn}X^n \ , \\ f_{pq} &= \epsilon_{pq}{}^m\Lambda_m + \Omega(v)\left(\delta_{pq}a^2 - \delta_{pm}X^m\delta_{qj}X^j\right) \ . \end{aligned} \tag{3.30}$$

Fixing the gauge as in (3.20), choosing $\Lambda_2 = 0$ (we are not able to gauge away Λ_3 once we have chosen (3.20)) and defining $x^2 = \Lambda_1{}^2 + \Lambda_3{}^2$ and $y = \Lambda_3$, we obtain the dual theory action

$$\begin{aligned} S^{dual}[v,x,y] =& S[v] + \frac{1}{4\pi\alpha'}\int d^2z\ \frac{1}{a^2\Omega(v)(x^2-y^2)}\ \left(a^4\Omega(v)^2\partial y\bar{\partial}y + x^2\partial x\bar{\partial}x\right) \\ &+ \frac{1}{4\pi}\int d^2z\ R^{(2)}\Phi' \ , \end{aligned} \tag{3.31}$$

where

$$\Phi' = \Phi - \log[a^2\Omega(v)\ (x^2-y^2)] \ . \tag{3.32}$$

To understand this metric better, we pass to new coordinates $x = \lambda$, $y = \lambda\cos\theta$. This coordinate system corresponds to the gauge choice $X^1 = 0$, $X^2 = a\sin\theta$, $X^3 = a\cos\theta$, $\Lambda_1 = \Lambda_2 = 0$ and $\lambda = \Lambda_3$. The dual action is now

$$\begin{aligned} S^{dual}[v,\theta,\lambda] = & S[v] \\ & + \frac{1}{4\pi\alpha'}\int d^2z\left\{a^2\Omega(v)\left(\partial\theta - \cot\theta\frac{\partial\lambda}{\lambda}\right)\left(\bar{\partial}\theta - \cot\theta\frac{\bar{\partial}\lambda}{\lambda}\right)\right. \\ & \qquad\qquad \left. + \frac{1}{a^2\Omega(v)\sin^2\theta}\partial\lambda\bar{\partial}\lambda\right\} \\ & + \frac{1}{4\pi}\int d^2z\ R^{(2)}\ \Phi' \ , \end{aligned} \tag{3.33}$$

with

$$\Phi' = \Phi - \log[a^2\Omega(v)\ \lambda^2\sin^2\theta] \ . \tag{3.34}$$

It can be seen from (3.33) that a duality transformation with respect to $SO(3)$ of the round metric on a two sphere, gives the metric that we would have obtained if we had made a duality transformation only with respect to the $U(1)$ subgroup of $SO(3)$ which represents the invariance of the original metric under $\varphi \to \varphi + \text{constant}$, except for the fact that under duality with respect to $SO(3)$

$$d\theta \to d\theta - \cot\theta \frac{d\lambda}{\lambda} \ . \tag{3.35}$$

In fact, the dual theory to (3.18) for $N = 3$ with respect to the abelian isometry $\varphi \to \varphi + \text{constant}$ is

$$\begin{aligned} S^{dual}[v,\theta,\lambda] = S[v] + \frac{1}{4\pi\alpha'}\int d^2z \ \left(a^2\Omega(v)\partial\theta\bar{\partial}\theta + \frac{1}{a^2\Omega(v)\sin^2\theta}\ \partial\lambda\bar{\partial}\lambda\right) \\ + \frac{1}{4\pi}\int d^2z\ R^{(2)}\Phi' \ , \end{aligned} \tag{3.36}$$

where the new dilaton in this case is

$$\Phi' = \Phi - \log[a^2\Omega(v)\ \sin^2\theta] \ . \tag{3.37}$$

We can now ask the question of whether the isometries of the original metrics are preserved. The answer is that generically this is *not* the case, that is, the original group of isomeries $\mathcal{G}$ is broken by the duality transformation. In our example for $SO(3)$ the original metric had three isometries. However (3.36) has only one and (3.33) has none as can be shown by solving directly the Killing equations. For the coordinate transformations $X^m \to X^m + \epsilon\zeta^m$ and v^μ invariant, the Killing equations in these metrics are

$$\begin{aligned} 0 &= \frac{\partial\zeta^m}{\partial v^\nu} g_{mn} \ , \\ 0 &= \frac{\partial\zeta^m}{\partial X^p} g_{mn} + \frac{\partial\zeta^m}{\partial X^n} g_{mp} + \zeta^m \frac{\partial g_{pn}}{\partial X^m} \ . \end{aligned} \tag{3.38}$$

The first equation means that the Killing vectors ζ^m do not depend on the coordinates v^μ. For the metric (3.36) the second equation gives

$$\begin{aligned} 0 &= \frac{\partial\zeta^\theta}{\partial\theta} \ , \\ 0 &= \frac{1}{a^2\Omega\ \sin^2\theta}\frac{\partial\zeta^\lambda}{\partial\theta} + a^2\Omega\ \frac{\partial\zeta^\theta}{\partial\lambda} \ , \\ 0 &= \frac{\partial\zeta^\lambda}{\partial\lambda} - \cot\theta\zeta^\theta \ , \end{aligned} \tag{3.39}$$

which have the unique solution

$$\begin{aligned} \zeta^\theta &= 0 \ , \\ \zeta^\lambda &= \text{constant} \ . \end{aligned} \tag{3.40}$$

That is, the only Killing vector of the metric in (3.36) is the Killing vector corresponding to constant shifts of λ. The Killing equations for the metric in (3.33) can also be solved explicitly since they are not much more complicated than equations (3.39). Actually, it is easier to do the calculation in the metric in (3.31) since it its diagonal. The Killing equations are in this case

$$\begin{aligned} 0 &= x\frac{\partial \zeta^x}{\partial x} - \frac{y^2}{(x^2-y^2)}\zeta^x + \frac{xy}{(x^2-y^2)}\zeta^y \ , \\ 0 &= a^2\Omega \ \frac{\partial \zeta^y}{\partial x} + \frac{x^2}{a^2\Omega}\frac{\partial \zeta^x}{\partial y} \ , \\ 0 &= \frac{\partial \zeta^y}{\partial y} - \frac{x}{(x^2-y^2)}\zeta^x + \frac{y}{(x^2-y^2)}\zeta^y \ , \end{aligned} \tag{3.41}$$

and it is easy to see that they have no solution. Therefore the metric in (3.31) (or (3.33)) has no continuous isometries (except of course for possible isometries of the metric $g_{\mu\nu}$ which have not been affected by the duality transformation with respect to the coordinates X^m). Notice however that (3.31) has the $\mathbb{Z}_2$ discrete isometries $x \to -x, y \to -y$ and $x \to x, y \to -y$. These symmetries belong to the original $O(3)$ transformations which are not connected to the identity and are left untouched by the proce ss of dualization.

4. Dual Geometries of 4-D Black Holes

We will now present, as a matter of illustration, some 4D black hole backgrounds and their duals. In order for the dual geometries to give string vacua, these geometries have to satisfy the string background equations. To lowest order in α' these equations are [22]

$$R_{MN} + D_M D_N \Phi - \frac{1}{4} H_M^{LP} H_{NLP} = 0 \tag{4.1}$$

$$D_L H^L_{MN} - (D_L \Phi) H^L_{MN} = 0 \tag{4.2}$$

$$R - 2\Lambda - (D\Phi)^2 + 2 D_M D^M \Phi - \frac{1}{12} H_{MNP} H^{MNP} = 0 \ , \tag{4.3}$$

where $\Lambda \equiv (c-26)/3$ is the cosmological constant in the effective string action, c is the central charge and, as usual $H_{MNP} \equiv \partial_{[M} B_{NP]}$. For the heterotic string, these equations have to be modified in order to include the background gauge fields.

Since any solution of Einstein's equations in vacuum is also a solution of (4.1)–(4.3) for constant dilaton Φ and antisymmetric tensor B_{MN}, we have at hand large classes of solutions of (4.1)–(4.3) with isometries [10]. In particular the Schwarzschild metric

$$ds^2 = -(1-2M/r)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{4.4}$$

times any CFT with $c = 22$ is a solution of (4.1)–(4.3) ($c = 6$ CFT is needed for the heterotic string, which can be obtained either by toroidal compactifications or Calabi–Yau spaces). The isometry group is given by time translations $t \to t + t_0$ together with the $SO(3)$ space rotations.

Direct application of the standard duality transformation to (4.4) for time translations, gives the dual metric

$$ds^2 = -\frac{dt^2}{1-2M/r} + \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{4.5}$$

with the dilaton field now given by $\Phi' = \Phi - \log(1-2M/r)$. This metric defines a geometry with naked singularities at $r = 0$ and $r = 2M$, as it can be verified by computing the curvature scalar $R = \frac{4M^2}{(2M-r)r^3}$. It is easy to check that equations (4.1)–(4.3) are satisfied by the dual metric and dilaton Φ'. We have then found a spherically symmetric solution of the string background equations, which is not a black hole, but has naked singularities and is dual to the Schwarzschild solution.

It is interesting to find the dual of the Schwarzschild black hole directly in Kruskal coordinates. This is done as for the 2D case described in the previous section. The dual geometry in these coordinates has the same singularity structure as in the dual of the Schwarzschild metric. This is different to the 2D case which is self dual. Therefore, even though the Kruskal diagrams for the 2D and 4D black holes are identical, their dual geometries are completely different.

Consider now the dual geometry of (4.4) with respect to the $SO(3)$ symmetry. We find

$$ds^2 = -(1-2M/r)dt^2 + \frac{dr^2}{1-2M/r} + \frac{1}{r^2(x^2-y^2)} \ [r^4 dy^2 + x^2 dx^2] \ , \tag{4.6}$$

with the new dilaton $\Phi' = \Phi - \log[r^2(x^2-y^2)]$. The regions $x = y$ and $r = 0$ are real singularities whereas $r = 2M$ is only a metric singularity corresponding to a horizon as in the original case. In fact, the curvature scalar is

$$R = -\frac{(2r^5 + 4Mx^2 - 4My^2 + 2ry^2)}{r^5(x^2-y^2)^2}. \tag{4.7}$$

Notice that the metric (4.6) is *not* spherically symmetric, in fact its only isometry is time translations. Neither is it asymptotically flat. For large r, the x dimension gets squeezed and the other dimensions behave like a $2+1$ dimensional space–time. The surfaces $x =$ constant are just $2+1$ dimensional black holes away from the singularities $\sin\theta = 0$, $(y = \pm x)$. Again, it is straightforward to check that this solution satisfies equations (4.1)–(4.3) thus providing new string vacua.

To find new solutions, we can certainly combine both dualities above. We can also consider different coordinate systems. For instance, using the Eddington-Finkelstein instead of the Schwarzschild coordinate system, we can find its dual with respect to time translations. It so happens that the dual metric is identical to (4.5), but now the new solution has non–trivial torsion. We have here come across another general feature of duality symmetries: string background solutions dual to geometries related by a coordinate transformation are not necessarily themselves related by a coordinate transformation . They all however give the same path integral.

A similar analysis can be done for the 4D charged dilatonic black holes of reference [23]. In this case the metric is

$$ds^2 = -\frac{(1-2M/r)}{(1-Q^2/Mr)}dt^2 + \frac{dr^2}{(1-2M/r)(1-Q^2/Mr)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{4.8}$$

the dilaton field is $\Phi = -\log(1-Q^2/Mr)$ and the electric field $F_{tr} = e^{\Phi}Q/(2r^2)$. It is very interesting to note that the dual of this solution with respect to time translations gives exactly the same solution except that the mass parameter M changes into $Q^2/2M$, therefore it relates the black hole domain $Q^2 < 2M^2$ to the naked singularity domain $Q^2 > 2M^2$. In particular, the extremal solution $Q^2 = 2M$ is selfdual. We have then discovered that 4D charged dilatonic black holes share a similar property with their 2D counterparts: a duality transformation can give the same geometry but interchange different regions. The 4D solutions however have more structure since another duality transformation coming from the $SO(3)$ isometry can be performed. As in the Schwarzschild case, this duality changes the angular part of the metric, while leaving the t, r components invariant:

$$\begin{aligned} ds^2 = -\frac{(1-2M/r)}{(1-Q^2/Mr)}dt^2 + &\frac{dr^2}{(1-2M/r)(1-Q^2/Mr)} \\ &+ \frac{1}{r^2(x^2-y^2)}\left[r^4 dy^2 + x^2 dx^2\right] \end{aligned} \tag{4.9}$$

with $\Phi' = -\log[(1-Q^2/Mr)r^2(x^2-y^2)]$ and invariant electric field. Again, the singularity structure in the r coordinate is the same as for the original metric whereas there are new singularities at $x=y$. Similar to the Schwarzschild dual, this solution is not asymptotically flat and for large r the geometry contracts to a $2+1$ space. The $x =$ constant surfaces are again $2+1$ dimensional black holes away from the singularities $y = \pm x$.

We should remark that all the backgrounds considered in this section are solutions of the leading order equations and then they are only approximate solutions of the exact field equations, valid in the domain of small curvature. Nevertheless, exact solutions with spherical symmetries surely exist for which the formalism presented in this paper should apply.

5. Conclusions

We have shown that there is a new type of duality symmetry in string theory associated with the existence of a non–abelian group of isometries on the target space–time in the worldsheet σ–model, which reduces to the usual target space duality when the isometry group is abelian. We have also shown that well known properties of target space duality do not extend to the general case. In particular, the group of isometries $\mathcal{G}$ is not preserved in general under a duality transformation. For instance, the dual background of a model with an abelian isometry also has an abelian isometry, which can be enhanced by a duality transfomation to a larger non–abelian group of isometries. This was the case of (3.36) which is dual to the Schwarzschild solution. The non–abelian case is even more dramatic since the original group of isometries can completely dissapear after dualization. Maybe the most interesting open problem that this work leaves is to find the mechanism by which duality transformation can be performed starting with backgrounds with no isometries such as Calabi–Yau spaces and gauged WZW models G/H for which the subgroup H is nonabelian [8,24].

There are several other ways in which our work could be extended. In [11], duality with respect to abelian isometries was formally proved to relate two different geometrical manifestations of a single conformal field theory. Probably, their analysis could be generalized to the non–abelian case considered in this paper. In particular, there are global considerations that must be studied when identifying the path integral in (3.4) with both dual theories. For example, for a world sheet with the topology of a torus, there are non trivial field configurations which should be considered. In order for the gauged theory to be equivalent to the original as conformal field theories, it was shown in [11] that in the abelian case, for compact isometries, the Lagrange multiplier should have a specific periodicity. In the non–abelian case, the non trivial field configurations ought to be considered too, though it is not known if there exists a range for the Lagrange multipliers which renders the theories equivalent. All we can say at the moment is that the dual geometries are not necessarily equivalent as conformal field theories but that they are related by an orbifold construction*.

The duality due to an abelian H is known to map Bianchi identities of one theory to field equations of the dual and viceversa. From this a set of transformations can be found [25,17,26] which *continuously* interpolates between Bianchi identities and field equations. This continuous symmetry, which is actually broken non–perturbatively, is what generates the moduli space to which two dual solutions belong. It is also very powerful when trying to find new solutions of the string background equations [27,28]. It will certainly be very interesting to investigate whether these continuous transformations exist in the non–abelian case. Also, at the moment we do not know what the full duality group is. Of course, it has to include the group of duality transformations with respect to *all* possible subgroups of $\mathcal{G}$ and therefore it must contain $SO(r, r, \mathbb{Z})$, where $r = \text{rank}\mathcal{G}$.

Finally, one of our motivations to study duality in 4D black holes, was to analyze the effects of duality on the singularities of those spaces. At the moment we have not been able to find a map that takes those singular regions to regular regions as was the case with the 2D black holes [5] and 3D black strings [6,8]. Instead there have appeared naked singularities. The origin of these new singularities could be very interesting to study in general. Also FRW cosmologies can be analyzed in the present approach and dual geometries to quasi realistic string cosmologies [29] could be investigated. We hope to discuss some of these issues in a future publication.

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